Computability and Transformers:
A Circuit Complexity Revival?

Michaël Cadilhac

Joint work with Andy Yang and David Chiang (Notre Dame)
Appearing at NeurIPS 2025

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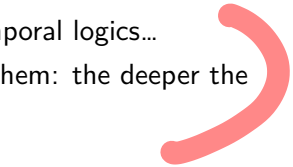

Outline

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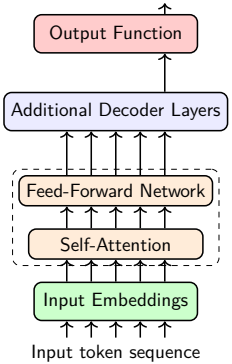

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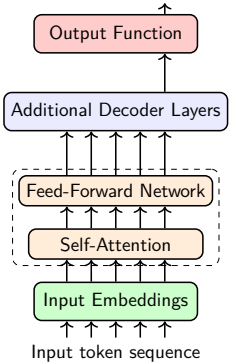

$E : \Sigma \to \mathbb{R}^d$

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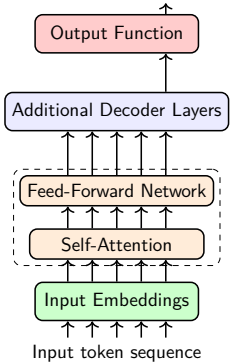

$$\text{lay}^{(0)}(i) = E(w_i) \in \mathbb{R}^d$$
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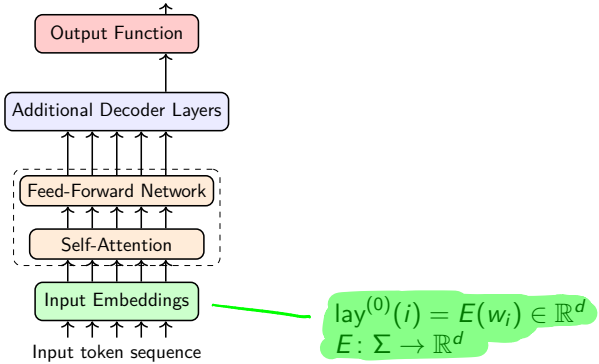

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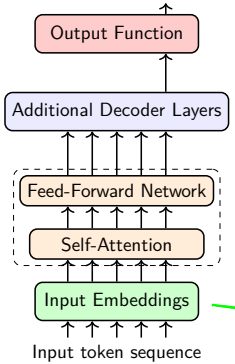

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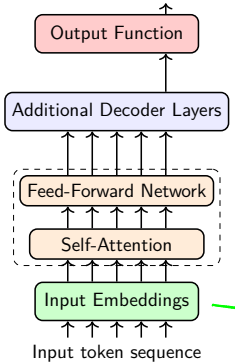

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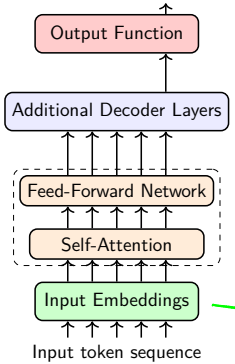

$$\mathrm{att}^{(1)}(i,j) = q^{(1)}(i) \cdot k^{(1)}(j) \in \mathbb{R}$$
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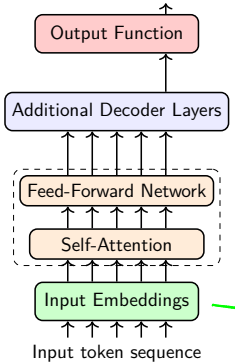

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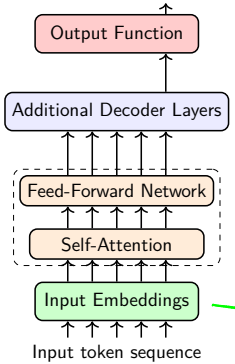

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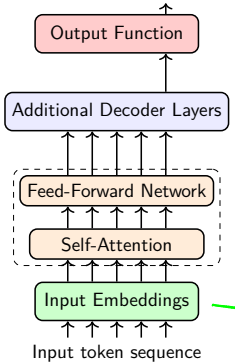

$$\text{pre-lay}^{(1)}(i) = \frac{\sum_{j \le i} \exp(\text{att}^{(1)}(i,j)) v^{(1)}(j)}{\sum_{j \le i} \exp(\text{att}^{(1)}(i,j))} \in \mathbb{R}^d$$

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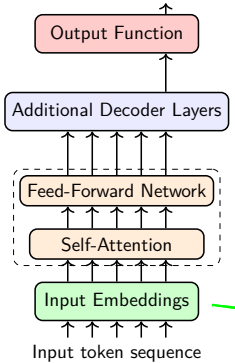

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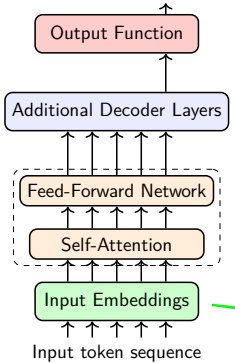

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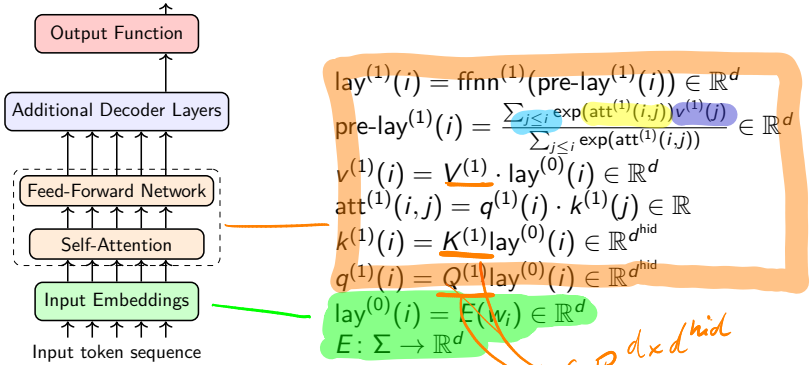

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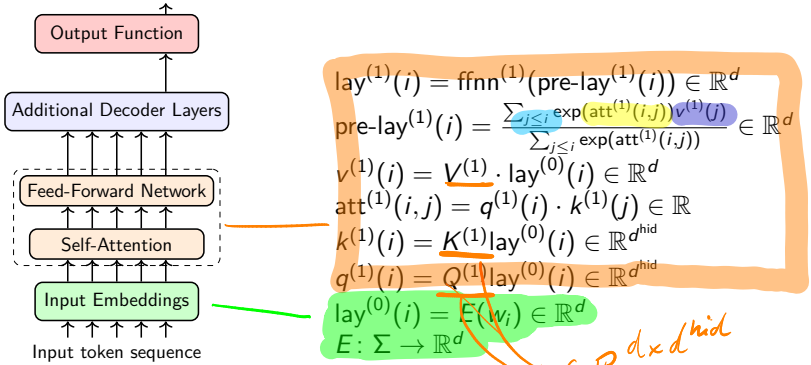

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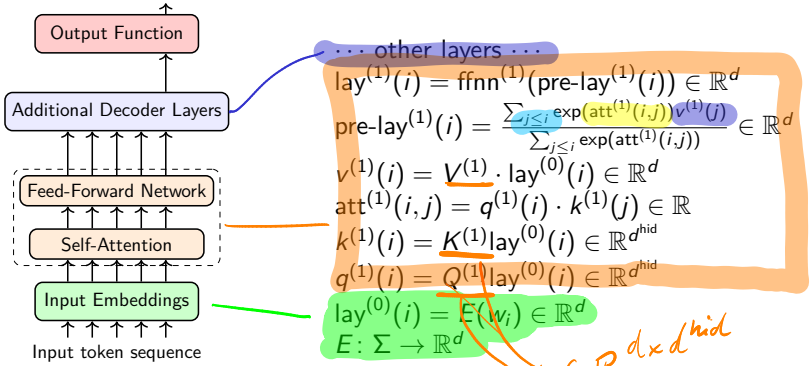

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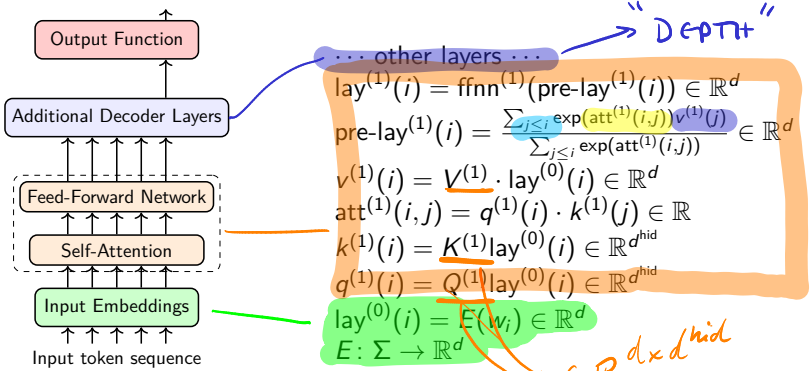

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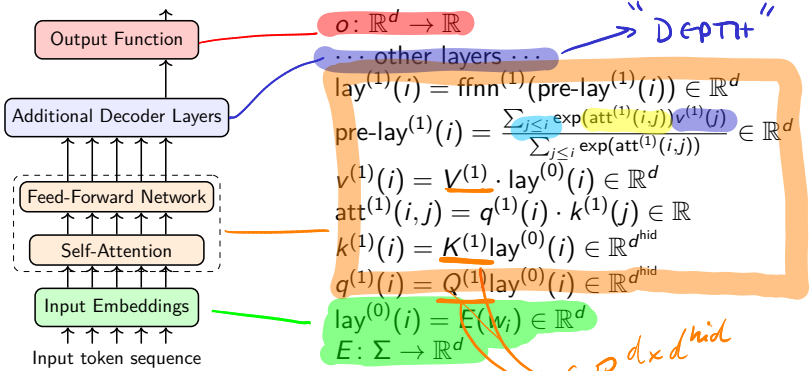

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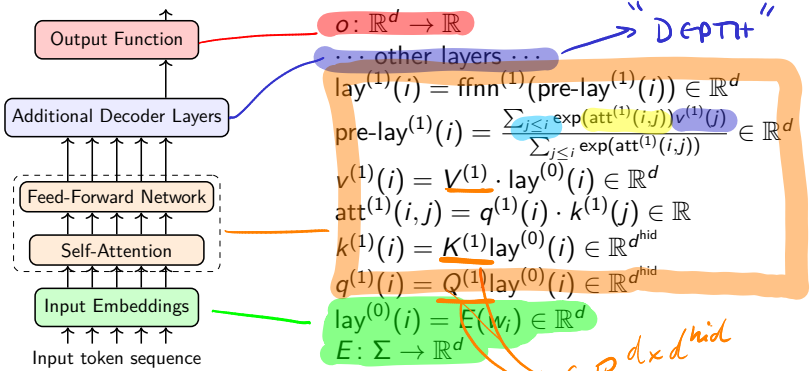

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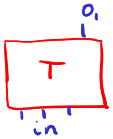

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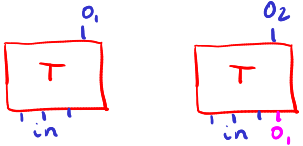

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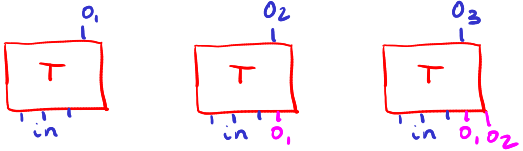

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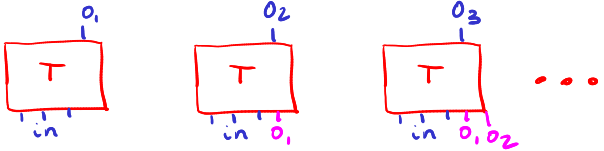

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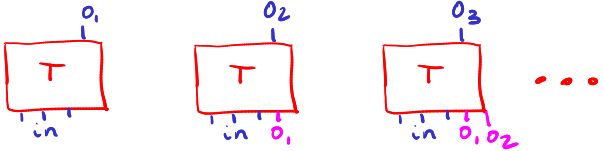

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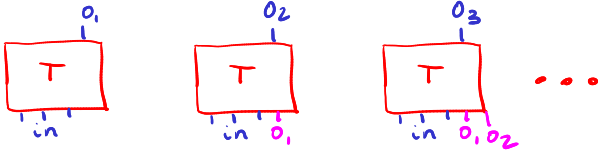

  - *Transformer language:* $\{w \in \Sigma^* \mid T(w) > 0\}$
  - *What is the complexity of these languages?*

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Chain of thought languages

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For each pos i, compute score against j, weigh by a value, then take avg (softmax)

Transformer

- Input encoding
- Attention layers/FFNN alternate

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Theorem (Pérez et al. 2019)

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- Track what is cell content at that position

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- After that, at each generation step, count number of left/right moves using attention
- Track what is cell content at that position
- Generate the next transition of the Turing machine □

Chain of thought languages

Theorem (Pérez et al. 2019)

"Transformers are Turing-complete"

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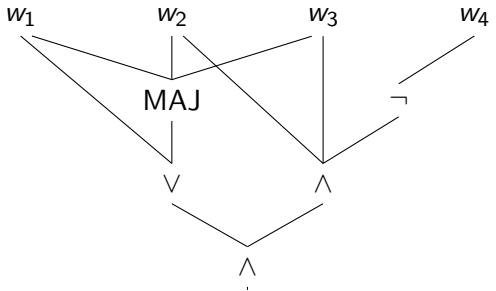
- Crucially relies on chain-of-thought!

Outline

Transformer languages

Preliminary 1: Circuit complexity



Transformer languages

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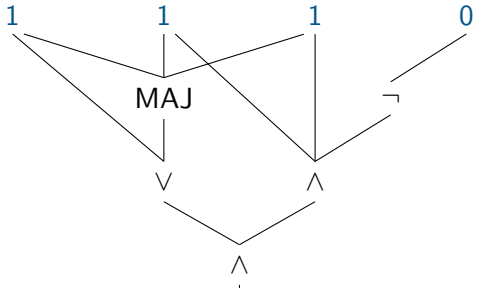

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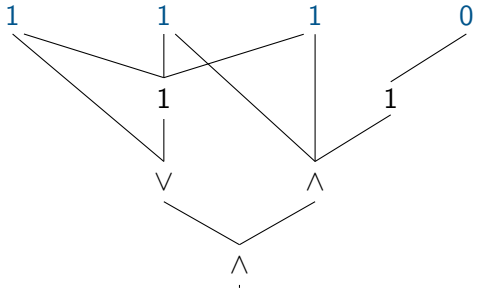

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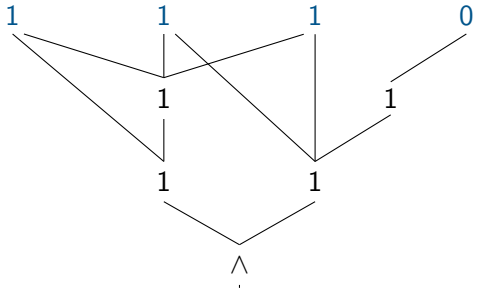

Attention layer

For each pos i, compute score against j, weigh by a value, then take avg (softmax)

Transformer

- Input encoding
- Attention layers/FFNN alternate

Transformer languages

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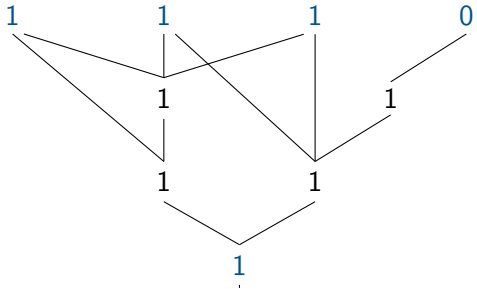

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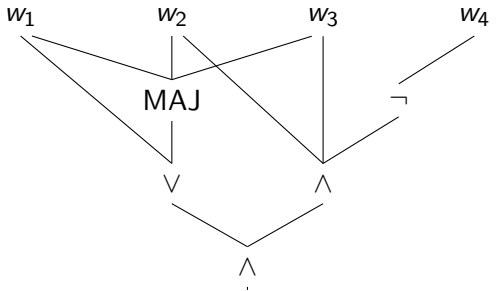

$$L_4 = \{1110, \ldots\}$$

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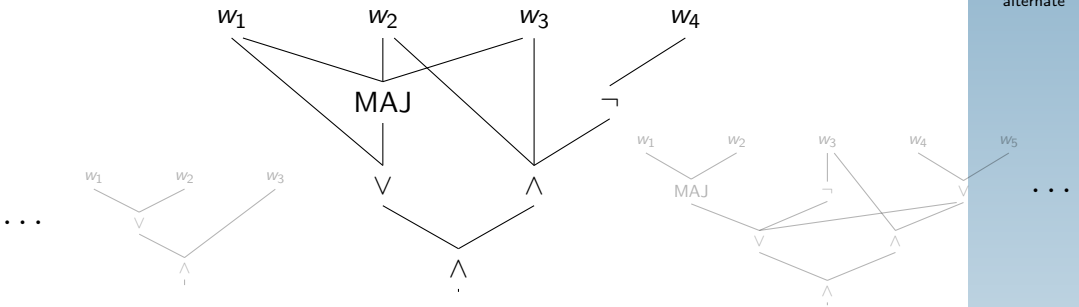

$L_4 = \{1110, \ldots\}$ $\quad L = \bigcup_{i \in \mathbb{N}} L_i$

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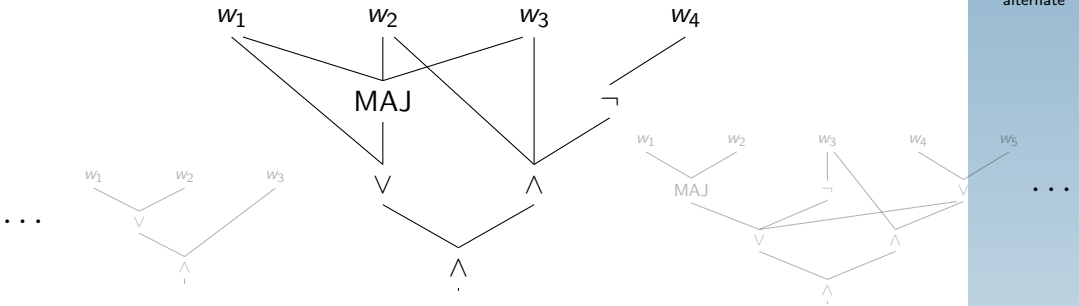
size(n) = number of gates



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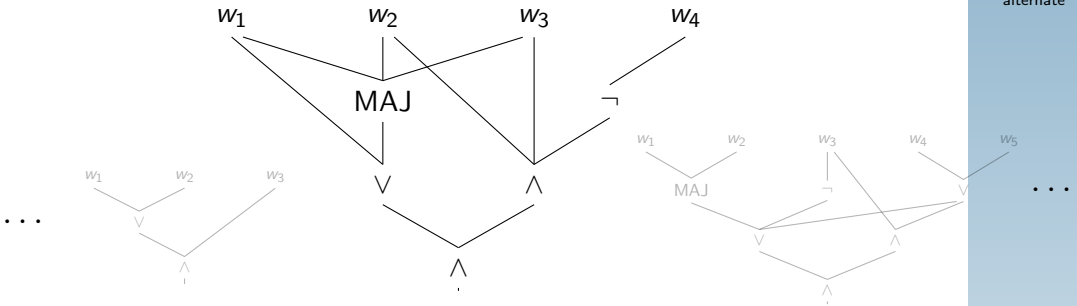

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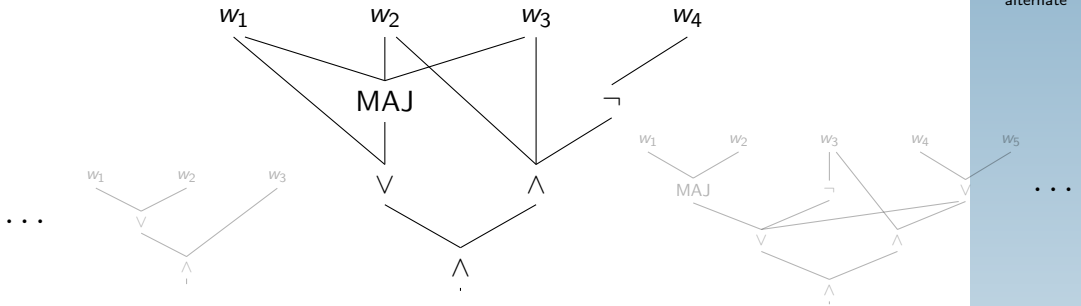

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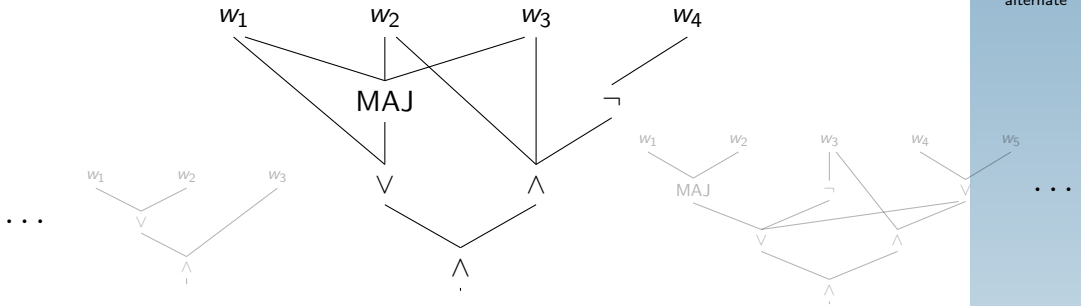

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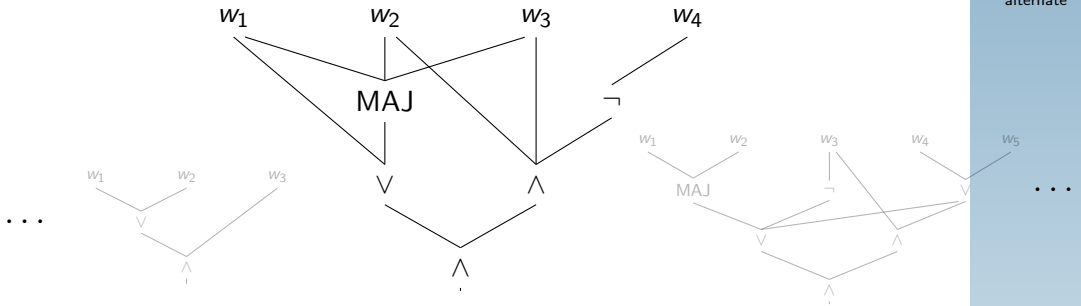

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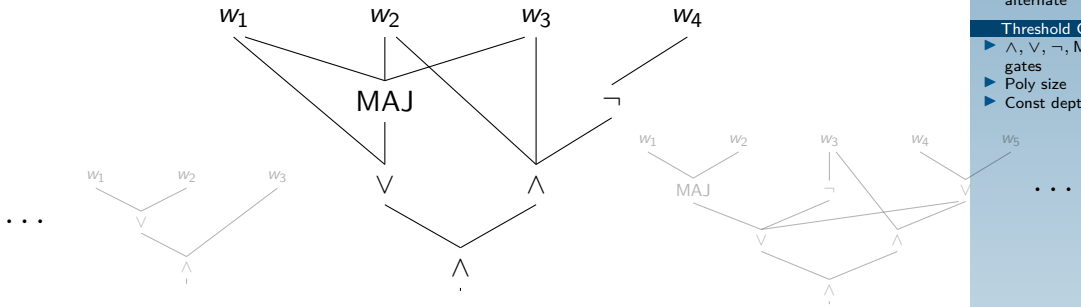

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Theorem (Merrill and Sabharwal, 2023)

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- What about a converse? Or something exact?

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Preliminary 2: Temporal logic with counting

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→ WELL PARENTHESIZED

a (b)

- TL[$\overleftarrow{\#}$]: temporal logic with $\overleftarrow{\#}$

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Proposition (Behle & Krebs 2008)

TL$[\overleftarrow{\#}]$ *languages are all implementable as threshold circuits of linear size*

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An exact characterization

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Theorem (Yang, Cadilhac, Chiang 2025)

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- The *main* lie: *positional encoding*
- Some other lesser lies (fixed precision at some places)

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- A reduction lemma: if L_m is in depth n, then L_{m-1} is in depth $n-1$ □

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A depth hierarchy

- L_k: all of words over $\{a, b\}$ that alternate k times between a's and b's

aaa bb aa $\in L_3$

Theorem (Yang, Cadilhac, Chiang 2025)

L_k is in depth-k $\mathrm{TL}[\overleftarrow{\#}]$ but not in depth-$(k-1)$ $\mathrm{TL}[\overleftarrow{\#}]$

Idea.

- Depth-1 $\mathrm{TL}[\overleftarrow{\#}]$ languages are commutative (letters commute)
- A reduction lemma: if L_m is in depth n, then L_{m-1} is in depth $n-1$ □

Corollary

Tranformers of depth k can express L_k but not L_{k+1}

Attention layer

For each pos i, compute score against j, weigh by a value, then take avg (softmax)

Transformer

- Input encoding
- Attention layers/FFNN alternate

Threshold Circuits

- $\wedge, \vee, \neg$, MAJ gates
- Poly size
- Const depth

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Transformer languages

A depth hierarchy in Practice: Expressing vs Training

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Accuracy on $[101, 150]$

depth →	1	2	3	4	5	6	7	8	9	10
L_3	100	100	100	100	100	100	100	100	100	100
L_4	44	100	100	100	100	100	100	100	100	100
L_5	42	67	100	100	100	100	100	100	100	100
L_6	36	46	61	100	100	100	100	100	100	100
L_7	36	52	55	65	100	100	100	100	100	100
L_8	30	37	42	42	65	100	100	100	100	100
L_9	8	35	40	46	43	38	87	75	100	80
L_{10}	5	8	25	33	25	27	44	30	44	47
L_{11}	5	10	12	17	16	17	24	18	24	17
L_{12}	0	0	0	9	13	0	4	4	19	16

Accuracy on $[151, 200]$

depth →	1	2	3	4	5	6	7	8	9	10
L_3	100	100	100	100	100	100	100	100	100	100
L_4	37	100	100	100	100	100	100	100	100	100
L_5	30	60	100	100	100	100	100	100	100	100
L_6	19	30	54	98	98	98	98	99	98	100
L_7	26	34	35	48	100	99	99	98	98	99
L_8	17	16	18	16	40	99	100	99	99	100
L_9	3	14	11	18	19	32	80	63	99	69
L_{10}	2	6	5	9	7	5	13	8	21	30
L_{11}	2	3	3	4	4	3	4	3	5	3
L_{12}	0	0	0	3	6	0	2	2	11	6

Accuracy on $[201, 250]$

depth →	1	2	3	4	5	6	7	8	9	10
L_3	99	100	100	100	100	100	100	100	100	100
L_4	26	99	99	100	100	100	100	100	100	100
L_5	24	50	99	99	98	100	100	100	100	100
L_6	8	15	38	93	89	92	90	94	91	96
L_7	18	20	19	31	96	93	96	89	92	95
L_8	10	7	8	7	23	97	98	96	97	98
L_9	1	7	3	7	9	26	75	60	98	62
L_{10}	0	7	1	2	1	1	3	2	7	22
L_{11}	0	0	1	1	1	0	1	0	1	0
L_{12}	0	0	0	0	0	0	0	0	1	1

Accuracy on $[251, 300]$

depth →	1	2	3	4	5	6	7	8	9	10
L_3	99	100	100	100	100	100	100	100	100	100
L_4	20	96	96	99	100	100	100	100	100	99
L_5	19	41	97	96	91	99	99	99	100	100
L_6	5	8	27	84	77	83	78	85	80	90
L_7	13	13	12	23	87	84	89	72	82	88
L_8	5	2	2	2	13	91	94	91	89	94
L_9	1	4	1	3	5	25	71	58	95	57
L_{10}	0	4	0	0	0	5	1	0	3	20
L_{11}	0	0	0	0	0	0	0	0	0	0
L_{12}	0	0	0	0	0	0	0	0	0	0

Figure 4: Experimental results. Corollary 5.2 predicts that a transformer with depth k can recognize language L_{k+2} but not L_{k+3} (demarcated by the black line). Up to L_9, this closely predicts our experimental results (shown as numbers and colors).

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Outline

Transformers

What can transformers compute, if we give them time to think?

What can transformers compute in one evaluation?

Conclusion

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We saw:

- Transformers = just a bunch of (mostly) linear operations

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- Main question: Learnability vs expressiveness

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> **On the circuit lower bound barrier.** It was suggested in [Hah20] that unconditional lower bounds against *encoder-only* transformer would imply breakthrough circuit lower bounds against linear threshold circuits.[1] Our result avoids this barrier by exploiting the information bottleneck and autoregressive nature of *decoder-only* models. Namely, our proof crucially depends on the fact that in decoder-only models, each token cannot attend to any token after it.

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